# Schwarzenberger bundles of arbitrary rank on the projective space

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### Abstract

We introduce a generalized notion of Schwarzenberger bundle on the projective space. Associated to this more general definition, we give an ad hoc notion of jumping subspaces of a Steiner bundle on $\mathbb{P}^n$ (which in rank $n$ coincides with the notion of unstable hyperplane introduced by Vallès, Ancona and Ottaviani). For the set of jumping hyperplanes, we find a sharp bound for its dimension. We also classify those Steiner bundles whose set of jumping hyperplanes have maximal dimension and prove that they are generalized Schwarzenberger bundles.

## Introduction

In [**6**], Schwarzenberger constructed some particular vector bundles $F$ of rank $n$ in the projective space $\mathbb{P}^n$, related to the secant spaces to rational normal curves and having a resolution of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus s} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus t} \longrightarrow F \longrightarrow 0.$$

Arbitrary vector bundles on $\mathbb{P}^n$ admitting such a resolution and having arbitrary rank (necessarily at least $n$) has been widely studied since then. These general bundles were called *Steiner bundles* by Dolgachev and Kapranov [**3**], because of their relation with the classical Steiner construction of rational normal curves. In that paper, the authors relate some Steiner bundles of rank $n$ (the so-called logarithmic bundles) to configurations of hyperplanes in $\mathbb{P}^n$. In fact, to a general configuration of $k$ hyperplanes they assign a Steiner bundle and, if this is not a Schwarzenberger bundle, there is a Torelli-type result in the sense that the configuration of hyperplanes can be reconstructed from the bundle (this is proved in [**3**] only for $k \geqslant 2n + 3$, and in general by Vallès [**9**]).

The result of Vallès and other related results by him and Ancona and Ottaviani (see [**1**]) are based on considering special hyperplanes associated to Steiner bundles of rank $n$, the so-called unstable hyperplanes. In particular, they prove that a Steiner bundle of rank $n$ is one of those constructed by Dolgachev and Kapranov if and only if it possesses at least $t + 1$ unstable hyperplanes (see [**1**, Corollary 5.4]) and if it has at least $t + 2$ unstable hyperplanes then it is a Schwarzenberger bundle and the set of unstable hyperplanes forms a rational normal curve (see [**9**, Théorème 3.1]). Hence, except in the last case, one recovers the original configuration of hyperplanes from its corresponding Steiner bundle. On the other hand, it is also true that, starting from a rational normal curve instead of a finite number of hyperplanes and constructing its corresponding Schwarzenberger bundle, one can still reconstruct the rational normal curve from the set of unstable hyperplanes.

The starting point of this paper is the last of the above results, that is, the correspondence between Schwarzenberger bundles and rational normal curves. First we introduce a generalized notion of Schwarzenberger bundle, which will be a Steiner bundle (of rank arbitrarily large)

obtained from a triplet $(X, L, M)$, where $X$ is any projective variety, and $L, M$ are globally generated vector bundles on $X$ of respective ranks $a$ and $b$ (see Example 1.5). In this context, the original vector bundles constructed by Schwarzenberger are those obtained from triplets in which $X = \mathbb{P}^1$, and $L, M$ are line bundles on $\mathbb{P}^1$. Independently, Vallès in [**10**] has recently given a similar definition in the case $a = b = 1$, assuming that $X$ is a curve, $M$ is very ample and $H^1(L \otimes M^{-1}) = 0$, but he allows $L$ to be just a coherent sheaf (so that $F$ is just a coherent sheaf, not necessarily locally free). He also generalizes the notion of logarithmic bundles to arbitrary rank and extends the Torelli-type results for configurations of lines in $\mathbb{P}^2$.

The first main problem we want to study is the following.

QUESTION 0.1.   When is a Steiner bundle a generalized Schwarzenberger bundle?

In order to answer this question, one needs to see whether it is possible to associate a triplet $(X, L, M)$ to a given Steiner bundle. Following the main ideas in [**1**, **3**, **9**], we observe that, for Schwarzenberger bundles, any point of $X$ yields a special subspace of $\mathbb{P}^n$, which we call $(a, b)$-jumping subspace (in fact we will introduce the more natural notion of jumping pair). This notion generalizes the notion of unstable hyperplane in [**1**, **9**] (with our definition, a jumping hyperplane is a hyperplane $H$ such that $h^0(F^*_{|H}) > h^0(F^*)$), so that we naturally wonder about the following Torelli-type problem.

QUESTION 0.2.   For which triplets $(X, L, M)$ does it happen that all the jumping subspaces come from points of $X$?

In this paper, we give a positive answer to Questions 0.1 and 0.2 when $a = b = 1$ and the set of jumping subspaces (which in this case are hyperplanes), or more generally the set of jumping pairs, has maximal dimension. More precisely, when $a = b = 1$ we first provide a sharp bound for the dimension of the set of jumping pairs of Steiner bundles (Theorem 2.8). Then we classify all Steiner bundles for which the set of jumping pairs has maximal dimension, showing that in all cases they are generalized Schwarzenberger bundles and that the variety $X$ in the triplet is obtained from the set of jumping pairs (Theorem 3.7).

I want to stress the fact that, despite the apparently abstract notions developed in the paper, most of the inspiration and techniques come from classical projective geometry (varieties of minimal degree, Segre varieties, linear projections, etc.).

The paper is structured as follows. In Section 1, we recall the main properties of Steiner bundles and introduce our generalized notion of Schwarzenberger bundle. We present four examples of Schwarzenberger bundles and prove (Proposition 1.11) that, in rank $n$, our definition coincides with the original Schwarzenberger bundles.

In Section 2, we introduce the notion of $(a, b)$-jumping subspaces and pairs of a Steiner bundle. In the particular case $a = b = 1$, we show (Theorem 2.8) that the set of jumping pairs has dimension at most $t - n - s + 1$ and that, if $n = 1$ or $s = 2$, any Steiner bundle is a Schwarzenberger bundle (thus generalizing to our general context the known result for rank $n$).

Finally, in Section 3, we classify Steiner vector bundles whose set of jumping pairs has maximal dimension (Theorem 3.7), showing that, in this case, they are Schwarzenberger bundles, precisely the examples introduced in Section 1. We include, as a first application of our theory, an improvement (Corollary 3.9) for line bundles of a result of Re (see [**5**]) about the multiplication map of sections. We finish with some remarks about the difficulty of the case of arbitrary $a, b$, and with some possible generalization of our definition to arbitrary varieties.

## 1. *Generalized Schwarzenberger bundles*

*General notation.* We will always work over a fixed algebraically closed ground field $k$. We will use the notation that, for a vector space $V$ over $k$, the projective space $\mathbb{P}(V)$ will be the set of hyperplanes of $V$ or equivalently the set of lines in the dual vector space $V^*$. If $v$ is a nonzero vector of $V^*$, we will write $[v]$ for the point of $\mathbb{P}(V)$ represented by the line $\langle v \rangle$ spanned by $v$. On the other hand, we will denote by $G(r, V)$ the Grassmann variety of $r$-dimensional subspaces of a vector space $V$.

Recall first the definition of Steiner bundle, in which we will include for convenience the invariants of the resolution.

DEFINITION. An $(s, t)$-*Steiner bundle* over $\mathbb{P}^n$ is a vector bundle $F$ with a resolution

$$0 \longrightarrow S \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow T \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow F \longrightarrow 0,$$

where $S$ and $T$ are vector spaces over $k$ of respective dimensions $s$ and $t$ (observe that the rank of $F$ is thus $t - s$).

REMARK 1.1. We recall from [3] the geometric interpretation of the resolution of a Steiner bundle. A morphism $\mathcal{O}_{\mathbb{P}^n}(-1) \to T \otimes \mathcal{O}_{\mathbb{P}^n}$ is equivalent to fixing an $(n+1)$-codimensional linear subspace $\Lambda \subset \mathbb{P}(T)$ and identifying $\mathbb{P}^n$ with the set, which we denote by $\mathbb{P}(T)^*_\Lambda$, of hyperplanes of $\mathbb{P}(T)$ containing $\Lambda$. Therefore giving a morphism $S \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \to T \otimes \mathcal{O}_{\mathbb{P}^n}$ is equivalent to fixing $s$ linear subspaces $\Lambda_1, \ldots, \Lambda_s \subset \mathbb{P}^{t-1}$ of codimension $n+1$ with a common parametrization by $\mathbb{P}^n$ of the sets $\mathbb{P}(T)^*_{\Lambda_i}$ of hyperplanes in $\mathbb{P}^{t-1}$ containing these $\Lambda_i$. Hence the projectivization of the fibre of $F$ at any point $p \in \mathbb{P}^n$ is the linear space $\mathbb{P}(F_p) \subset \mathbb{P}(T)$ consisting of the intersection of the $s$ hyperplanes of $\mathbb{P}(T)^*_{\Lambda_1}, \ldots, \mathbb{P}(T)^*_{\Lambda_s}$ corresponding to $p$.

We recall in the next lemmas the standard characterization of Steiner bundles by means of linear algebra, and introduce the notation that we will use throughout the paper.

LEMMA 1.2. *Given vector spaces $S$ and $T$ over $k$, the following data are equivalent.*

(i) *A Steiner bundle $F$ with resolution $0 \to S \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \to T \otimes \mathcal{O}_{\mathbb{P}^n} \to F \to 0$.*

(ii) *A linear map $\varphi : T^* \to S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathrm{Hom}(H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*, S^*)$ such that, for any $u \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*$ and any $v \in S^*$, there exists $f \in \mathrm{Hom}(H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*, S^*)$ in the image of $\varphi$ satisfying $f(u) = v$.*

*Proof.* Taking duals, giving a morphism $S \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \to T \otimes \mathcal{O}_{\mathbb{P}^n}$ is equivalent to giving a morphism

$$\psi : T^* \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow S^* \otimes \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}(-1), S^* \otimes \mathcal{O}_{\mathbb{P}^n})$$

and this is clearly equivalent to giving linear map

$$\varphi : T^* \longrightarrow H^0(S^* \otimes \mathcal{O}_{\mathbb{P}^n}(1)) = S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathrm{Hom}(H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*, S^*).$$

Hence we need to characterize when the morphism $\psi$ induced by $\varphi$ is surjective, that is, when the fibres of $\psi$ are surjective at any point of $\mathbb{P}^n$. To this purpose, we observe that, for any point $[u] \in \mathbb{P}^n$ corresponding to a nonzero vector $u \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*$, the fibre of $\psi$ at $[u]$ is the linear map $T^* \to \mathrm{Hom}(\langle u \rangle, S^*)$ consisting of the restriction of $\varphi$. Hence this map is surjective if and only if, for any $v \in S^*$, there exists $f \in \mathrm{Hom}(H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*, S^*)$ in the image of $\varphi$ satisfying $f(u) = v$. This proves the lemma. $\square$

LEMMA 1.3.   *With the notation of Lemma 1.2, the following data are equivalent.*
  (i)  *A linear subspace $K \subset T^*$ contained in the kernel of $\varphi$.*
  (ii)  *An epimorphism $F \to K^* \otimes \mathcal{O}_{\mathbb{P}^n}$.*
  (iii)  *A splitting $F = F_K \oplus (K^* \otimes \mathcal{O}_{\mathbb{P}^n})$.*
*In this case, $F_K$ is the Steiner bundle corresponding, by Lemma 1.2, to the natural map $T^*/K \to S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$. As a consequence, if $T_0^*$ is the image of $\varphi$ and $F_0$ is the Steiner bundle corresponding to the inclusion $T_0^* \to S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, then $H^0(F_0^*) = 0$ and $F = F_0 \oplus (T/T_0) \otimes \mathcal{O}_{\mathbb{P}^n}$. In particular, $H^0(F^*) = 0$ if and only if $\varphi$ is injective.*

*Proof.*   The equivalence of (ii) and (iii) comes from the fact that $F$ is generated by its global sections. In the situation of (i), we have a map $\bar{\varphi} : T^*/K \to S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ which, by Lemma 1.2, induces a Steiner bundle $F_K$. We clearly have a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S \otimes \mathcal{O}_{\mathbb{P}^n}(-1) & \longrightarrow & (T^*/K)^* \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & F_K & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S \otimes \mathcal{O}_{\mathbb{P}^n}(-1) & \longrightarrow & T \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K^* \otimes \mathcal{O}_{\mathbb{P}^n} & = & K^* \otimes \mathcal{O}_{\mathbb{P}^n} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

induced by the first two rows, so that the last column yields situation (ii). Reciprocally, given an epimorphism $F \to K^* \otimes \mathcal{O}_{\mathbb{P}^n}$, the resolution of $F$ yields another epimorphism $T \otimes \mathcal{O}_{\mathbb{P}^n} \to K^* \otimes \mathcal{O}_{\mathbb{P}^n}$, so that we can consider $K$ as a subspace of $T^*$. We thus get a diagram as above, now induced by its last two rows. Dualizing the diagram and taking cohomology, we get that $\varphi : T^* \to S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ factorizes through $T^*/K$, so that $K$ is contained in the kernel of $\varphi$, which is situation (i). Observe finally that $F_0$ is nothing but $F_{\ker \varphi}$.                                                    $\square$

DEFINITION.   With the above notation, we will say that a Steiner bundle is *reduced* if $\varphi$ is injective, that is, if $H^0(F^*) = 0$. The Steiner bundle $F_0$ will be called the *reduced summand* of $F$.

REMARK 1.4.   Observe that, since there are not Steiner bundles on $\mathbb{P}^n$ of rank smaller than $n$ (see, for instance, [**3**, Proposition 3.9]), any Steiner bundle of rank $n$ must coincide with its reduced summand, and hence it is reduced. Notice also that the only reduced Steiner bundle with $s = 1$ is $T_{\mathbb{P}^n}(-1)$. This is why we will consider only the cases $s \geqslant 2$.

Our generalized notion of Schwarzenberger bundle will come from the following example, in which we will use a slightly more general framework.

EXAMPLE 1.5.   Let $X$ be a projective variety and consider two coherent sheaves $L$ and $M$ on $X$, and assume $L$ is locally free. If $h^0(M) = n + 1$, we identify $\mathbb{P}^n$ with $\mathbb{P}(H^0(M)^*)$, the set of lines in $H^0(M)$. Consider the natural composition

$$
H^0(L) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow H^0(L) \otimes H^0(M) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow H^0(L \otimes M) \otimes \mathcal{O}_{\mathbb{P}^n}.
$$

For each nonzero $\sigma \in H^0(M)$, the fibre of the above composition at the point $[\sigma] \in \mathbb{P}^n$ is

$$
H^0(L) \otimes \langle \sigma \rangle \longrightarrow H^0(L) \otimes H^0(M) \longrightarrow H^0(L \otimes M).
$$

and, identifying $H^0(L) \otimes \langle \sigma \rangle$ with $H^0(L)$, we get that the composition is injective since it can be identified with $H^0(L) \xrightarrow{\cdot \sigma} H^0(L \otimes M)$. We thus have a Steiner vector bundle $F$ defined as a cokernel

$$0 \longrightarrow H^0(L) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow H^0(L \otimes M) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow F \longrightarrow 0.$$

Observe that the map $\varphi$ of Lemma 1.2 is, in this case, the dual of the multiplication map $H^0(L) \otimes H^0(M) \to H^0(L \otimes M)$. In particular, $F$ is reduced if and only if this multiplication map is surjective. More generally, according to Lemma 1.3, if $W$ is the image of the multiplication map, we have $F = F_0 \oplus (H^0(L \otimes M)/W) \otimes \mathcal{O}_{\mathbb{P}^n}$, where $F_0$ is the reduced summand of $F$.

DEFINITION. Let $X$ be a projective variety, and let $L$ and $M$ be globally generated vector bundles on $X$. The *Schwarzenberger bundle* of the triplet $(X, L, M)$ will be the Steiner vector bundle constructed in Example 1.5.

REMARK 1.6. By Remark 1.1, the geometry of a Schwarzenberger bundle $F$ when $L$ and $M$ are line bundles is related to the geometry of the map $\varphi_{L \otimes M} : X \to \mathbb{P}(H^0(L \otimes M))$ defined by $L \otimes M$. Indeed, in this case, $\mathbb{P}^n$ is identified with the complete linear series $|M|$ of effective divisors on $X$. For each $D \in |M|$, Example 1.5 shows that the fibre $F_D$ is the cokernel of the map $H^0(L) \to H^0(L \otimes M)$ defined by a section of $M$ vanishing at $D$. Hence the projectivization $\mathbb{P}(F_D) \subset \mathbb{P}(H^0(L \otimes M))$ is the linear span of the divisor $D$ regarded as a subset in $\mathbb{P}(H^0(L \otimes M))$ via $\varphi_{L \otimes M}$. Thus Remark 1.1 is saying that the set of these linear spans can be constructed by fixing linear subspaces $\Lambda_1, \ldots, \Lambda_s \subset \mathbb{P}(H^0(L \otimes M))$, defining common parametrizations of the $\mathbb{P}(H^0(L \otimes M))^*_{\Lambda_i}$ and taking the intersection of corresponding hyperplanes.

Therefore, when considering only Schwarzenberger bundles coming from line bundles, Question 0.1 can be stated geometrically as follows. *Given $s$ linear subspaces $\Lambda_1, \ldots, \Lambda_s \subset \mathbb{P}(T)$ of codimension $n + 1$ such that the $\mathbb{P}(T)^*_{\Lambda_i}$ are parametrized by the same $\mathbb{P}^n$, do the intersections of the corresponding hyperplanes describe the span of the divisors of some complete linear system of a variety?*

We give now four representative examples of Schwarzenberger bundles.

EXAMPLE 1.7. When $(X, L, M) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s - 1), \mathcal{O}_{\mathbb{P}^1}(n))$, one obtains an $(s, s + n)$-Steiner bundle of rank $n$, which is precisely the vector bundle constructed by Schwarzenberger. If $n = 1$, then Remark 1.1 provides, for any $(s, s + 1)$-Steiner bundle, the classical Steiner construction of the rational normal curve in $\mathbb{P}^s$, so that the answer to Question 0.1 is positive. On the other hand, as can be found, for example, in [**1**] or [**9**], if $s = 2$, then any $(2, n + 2)$-Steiner bundle is a Schwarzenberger bundle, while if $s > 2$ and $n > 1$, then a general $(s, s + n)$-Steiner bundle is not a Schwarzenberger bundle.

EXAMPLE 1.8. Let $F = \bigoplus_{i=1}^{t-s} \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geqslant 1$ for $i = 1, \ldots, t - s$, and assume $\deg F = a_1 + \ldots + a_{t-s} = s$. Write $X = \mathbb{P}(F)$ and let $\mathcal{O}_X(h)$ denote the tautological quotient line bundle (equivalently, $X$ is a smooth rational normal scroll $X \subset \mathbb{P}^{t-1}$ of dimension $t - s$ and degree $s$). If $f$ is the class of a fibre of the scroll, the positivity of the $a_i$ implies that $L := \mathcal{O}_X(h - f)$ is globally generated. Then, if $M = \mathcal{O}_X(f)$, the Schwarzenberger bundle of $(X, L, M)$ is an $(s, t)$-Steiner bundle on $\mathbb{P}^1$. By the geometric interpretation given in Remark 1.6, the fibre of this Schwarzenberger bundle at any point of $\mathbb{P}^1$ is nothing but the corresponding fibre of the scroll $X$. Therefore, this Schwarzenberger bundle is precisely the original $F$. This

shows that any ample vector bundle on $\mathbb{P}^1$ is a Schwarzenberger bundle. Observe that $F$ can also be regarded as the Schwarzenberger bundle of the triplet $(\mathbb{P}^1, F(-1), \mathcal{O}_{\mathbb{P}^1}(1))$.

We consider next the symmetric example with respect to the previous one, by just permuting $L$ and $M$. Observe that, even if this permutation produces different vector bundles (in fact defined on different projective spaces), most of our results on Steiner bundles will keep some symmetry of this type (for example, in Theorem 2.8 the roles of $n+1$ and $s$ are symmetric). As the referee remarked to us, this symmetry is the so-called Gale transform.

EXAMPLE 1.9.   Let $X$ be a smooth rational normal scroll $X \subset \mathbb{P}^{t-1}$ of dimension $t-n-1$ and degree $n+1$ defined by $E = \bigoplus_{i=1}^{t-n-1} \mathcal{O}_{\mathbb{P}^1}(a_i)$, so that $a_i \geqslant 1$ for $i = 1, \ldots, t-n+1$ and $\sum_{i=1}^{t-n+1} a_i = n+1$. Let $h$ and $f$ denote, respectively, the class of a hyperplane and a fibre of the scroll. Then, if $L = \mathcal{O}_X(f)$ and $M = \mathcal{O}_X(h-f)$, the Schwarzenberger bundle $F$ of $(X, L, M)$ is a $(2,t)$-Steiner bundle. We will see in Theorem 2.8(iv) that in this case any $(2,t)$-Steiner bundle is obtained in this way (the case $t = n+2$ is exactly the case $s = 2$ of Example 1.7). As before, $F$ can also be regarded as the Schwarzenberger bundle of the triplet $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), E(-1))$.

EXAMPLE 1.10.   The Schwarzenberger bundle of the triplet $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1))$ is a $(3,6)$-Steiner bundle $F$ of rank 3 over $\mathbb{P}^2$. If we identify this last $\mathbb{P}^2$ with the set of conics of the Veronese surface $V \subset \mathbb{P}^5$, then the projectivization of the fibre of $F$ at the element of $\mathbb{P}^2$ corresponding to a conic $C \subset V$ gives the plane of $\mathbb{P}^5$ spanned by $C$. In fact, it follows that $F = S^2(T_{\mathbb{P}^2}(-1))$ (see [**2**, p. 615]), so that $F_{|L} = \mathcal{O}_L \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(2)$ for any line $L \subset \mathbb{P}^2$. We will see in Remark 2.6 that a general $(3,6)$-Steiner bundle is not obtained in this way.

We end this section by reformulating in terms of our generalized Schwarzenberger bundles the results of Re about the multiplication map for vector bundles (we will improve his results in Corollary 3.9 in the case of rank 1). This will imply in particular that our generalized Schwarzenberger bundles of rank $n$ are exactly those constructed originally by Schwarzenberger:

PROPOSITION 1.11.   *Let $F$ be an $(s,t)$-Steiner bundle on $\mathbb{P}^n$ that is the Schwarzenberger bundle of a triplet $(X, L, M)$, with $\mathrm{rk}(L) = a$ and $\mathrm{rk}(M) = b$. Then we have the following.*

(i)   $t \geqslant bs + a(n+1) - ab$.
(ii)   *If equality holds in* (i), *then $F$ is the Schwarzenberger bundle of a triplet $(\mathbb{P}^1, L, M)$, where $\deg(L) = s - a$ and $\deg(M) = n + 1 - b$.*
(iii)   *Any Schwarzenberger bundle of rank $n$ is as in Example* 1.7.

*Proof.*   By [**5**, Theorem 1] we have $h^0(L \otimes M) \geqslant bh^0(L) + ah^0(M) - ab$, which is inequality (i). Moreover, Theorem 2 of [**5**] says that, when the above inequality is an equality, then there exist a map $f : X \to \mathbb{P}^1$ and vector bundles $L'$ and $M'$ on $\mathbb{P}^1$ such that $L = f^*L'$, $H^0(L) = f^*H^0(L')$, $M = f^*M'$ and $H^0(M) = f^*H^0(M')$. This means that $F$ is also the Schwarzenberger bundle of the triplet $(\mathbb{P}^1, L', M')$. This proves (ii), since the Riemann–Roch theorem for vector bundles on $\mathbb{P}^1$ implies $s = \deg(L') + a$ and $n + 1 = \deg(M') + b$.

In order to prove (iii), observe that $F$ has rank $n$ if and only if $t = h^0(L \otimes M) = h^0(L) + h^0(M) - 1$. Since $L$ and $M$ are globally generated, it follows that $h^0(L) \geqslant a$ and $h^0(M) \geqslant b$.

Therefore

$$
\begin{aligned}
t - bs - a(n+1) + ab &= (h^0(L) + h^0(M) - 1) - bh^0(L) - ah^0(M) + ab \\
&= -(b-1)h^0(L) - (a-1)h^0(M) + ab - 1 \\
&\leqslant (b-1)a + (a-1)b + ab - 1 = -(a-1)(b-1) \leqslant 0.
\end{aligned}
$$

By (i) we have that all inequalities are equalities and in particular $a = b = 1$, and by (ii) we also have that $F$ is the Schwarzenberger bundle of a triplet $(\mathbb{P}^1, L, M)$, where $L$ and $M$ are line bundles on $\mathbb{P}^1$ of respective degrees $s - 1$ and $n$, from which the result follows. $\qquad\square$

## 2. Jumping subspaces of Steiner bundles

In order to answer Question 0.1, one needs to try to produce a triplet $(X, L, M)$ from a Steiner bundle $F$. The main idea to find a candidate for $X$ comes from the fact that, since $M$ is a globally generated vector bundle of rank $b$, any point $x \in X$ yields a $b$-codimensional subspace $H^0(M \otimes \mathcal{J}_x) \subset H^0(M)$ consisting of the sections of $M$ vanishing at $x$. Thus the points of $X$ give particular linear subspaces of codimension $b$ in the projective space $\mathbb{P}^n = \mathbb{P}(H^0(M)^*)$ on which the Schwarzenberger bundle is defined. Hence our goal is to look for some special property of these linear subspaces for Schwarzenberger bundles and see whether, for an arbitrary Steiner bundle, the set of subspaces satisfying that property could play the role of $X$. This is the scope of the following.

LEMMA 2.1. *Let $F$ be a Steiner bundle over $\mathbb{P}^n$. Then we have the following.*
 (i) *For any nonempty linear subspace $\Lambda \subset \mathbb{P}^n$, there is a canonical commutative diagram*

$$
\begin{array}{ccccc}
S^* \otimes H^0(\mathcal{J}_\Lambda(1)) & \xrightarrow{\cong} & H^1(F^* \otimes \mathcal{J}_\Lambda) & & \\
\downarrow & & \downarrow{\scriptstyle\phi} & & \\
T^* \xrightarrow{\varphi} S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) & \longrightarrow & H^1(F^*) & \longrightarrow & 0.
\end{array}
$$

 (ii) *If $F$ is the Schwarzenberger bundle of the triplet $(X, L, M)$ and $\Lambda \subset \mathbb{P}^n$ is the subspace corresponding to $H^0(M \otimes \mathcal{J}_x) \subset H^0(M)$ for some $x \in X$, then there exists an $a$-dimensional linear subspace $A \subset S^*$ such that $A \otimes H^0(\mathcal{J}_\Lambda(1))$ is in the kernel of $\phi$.*

*Proof.* Diagram (i) comes by taking cohomology in the dual of the resolution of $F$ and its twist by $\mathcal{J}_\Lambda$. For (ii), if $F$ is the Schwarzenberger bundle of the triplet $(X, L, M)$, we have

$$
H^0(\mathcal{O}_{\mathbb{P}^n}(1)) = H^0(M)^*, \quad S = H^0(L), \quad T = H^0(L \otimes M)
$$

and $\varphi$ is the dual of the multiplication map $H^0(L) \otimes H^0(M) \to H^0(L \otimes M)$. Moreover, if $\Lambda$ is the linear subspace corresponding to $H^0(M \otimes \mathcal{J}_x) \subset H^0(M)$, for some $x \in X$, we also have $H^0(\mathcal{J}_\Lambda(1)) = H^0(M_x)^*$. It is clear that $\varphi$ maps $H^0(L_x \otimes M_x)^*$ isomorphically to $H^0(L_x)^* \otimes H^0(M_x)^*$. Hence, it follows that $H^0(L_x)^* \otimes H^0(\mathcal{J}_\Lambda(1))$ is mapped to zero in $H^1(F^*)$. $\qquad\square$

This suggests the following definition.

DEFINITION. Let $F$ be a Steiner bundle over $\mathbb{P}^n$. An $(a, b)$-*jumping subspace* of $F$ is a $b$-codimension subspace $\Lambda \subset \mathbb{P}^n$ satisfying that, with the identification given in (1), there exists an $a$-dimensional linear subspace $A \subset S^*$ such that $A \otimes H^0(\mathcal{J}_\Lambda(1))$ is in the kernel of the natural map $H^1(F^* \otimes \mathcal{J}_\Lambda) \to H^1(F^*)$. The pair $(A, \Lambda)$ will be called $(a, b)$-*jumping pair* of $F$. We will write $J_{a,b}(F)$ and $\tilde{J}_{a,b}(F)$ to denote, respectively, the set of $(a, b)$-jumping subspaces and the set of $(a, b)$-jumping pairs of $F$. We will also write $\Sigma_{a,b}(F)$ to denote the set of subspaces $A \subset S^*$

for which there exists a $b$-codimensional subspace $\Lambda \subset \mathbb{P}^n$ such that $(A, \Lambda)$ is an $(a, b)$-jumping subspace of $F$. A $(1, 1)$-jumping subspace or $(1, 1)$-jumping pair is called simply a *jumping hyperplane* or *jumping pair*, respectively, and we just write $J(F)$ or $\tilde{J}(F)$ to denote the set of jumping hyperplanes or jumping pairs of $F$, respectively. Similarly we write $\Sigma(F) := \Sigma_{1,1}(F)$.

We prove next a series of easy properties of jumping spaces and pairs.

LEMMA 2.2.   *Let $F$ be a Steiner bundle over $\mathbb{P}^n$. Then the following hold.*

(i) *For any $a, b$, the set of $(a, b)$-jumping pairs of $F$ coincides with the set of $(a, b)$-jumping pairs of its reduced summand $F_0$. In particular, $J_{a,b}(F) = J_{a,b}(F_0)$ and $\Sigma_{a,b}(F) = \Sigma_{a,b}(F_0)$.*

(ii) *If $A \subset S^*$ is a linear subspace of dimension $a$ and $\Lambda \subset \mathbb{P}^n$ is a subspace of codimension $b$, then $(A, \Lambda)$ is an $(a, b)$-jumping pair of $F$ if and only if $A \otimes H^0(\mathcal{J}_\Lambda(1))$ is in the image $T_0^*$ of $\varphi : T^* \to S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$.*

(iii) *Any $(a, b)$-jumping pair $(A, \Lambda)$ of $F$ induces, in a canonical way, a split quotient $F_{0|\Lambda} \to A^* \otimes H^0(\mathcal{J}_\Lambda(1))^* \otimes \mathcal{O}_\Lambda$.*

(iv) *If $b = 1$, then a hyperplane $H \subset \mathbb{P}^n$ is an $(a, 1)$-jumping subspace if and only if there is a quotient $F_0|H \to \mathcal{O}_H^{\oplus a}$, that is, $h^0(F_{|H}^*) \geqslant h^0(F^*) + a$.*

*Proof.*   Part (i) is obvious from the splitting (see Lemma 1.3) $F = F_0 \oplus (T/T_0) \otimes \mathcal{O}_{\mathbb{P}^n}$, so that the maps $H^1(F^* \otimes \mathcal{J}_\Lambda) \to H^1(F^*)$ and $H^1(F_0^* \otimes \mathcal{J}_\Lambda) \to H^1(F_0^*)$ are the same for any subspace $\Lambda$. Part (ii) follows at once from Lemma 2.1(i).

To prove (iii), let $(A, \Lambda)$ be a jumping pair of $F$. By (ii), this means that $A \otimes H^0(\mathcal{J}_\Lambda(1))$ can be regarded as a subspace of $T_0^*$. On the other hand, recall that $F_0$ is the Steiner bundle constructed (see Lemma 1.2) from the inclusion $T_0^* \to S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$. It is clear that $F_{0|\Lambda}$ is the Steiner bundle constructed from the composition

$$T_0^* \longrightarrow S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \longrightarrow S^* \otimes H^0(\mathcal{O}_\Lambda(1))$$

and, since $A \otimes H^0(\mathcal{J}_\Lambda(1))$ is contained in its kernel, Lemma 1.3 gives the wanted split quotient.

Finally, the 'only if' part of (iv) is (iii). Reciprocally, assume that there is a quotient $F_0|H \to \mathcal{O}_H^{\oplus a}$ for some hyperplane $H \subset \mathbb{P}^n$, which is equivalent, by the splitting $F = F_0 \oplus (T/T_0) \otimes \mathcal{O}_{\mathbb{P}^n}$, to the inequality $h^0(F_{|H}^*) \geqslant h^0(F^*) + a$. From the exact sequence

$$0 = H^0(F^* \otimes \mathcal{J}_H) \longrightarrow H^0(F^*) \longrightarrow H^0(F_{|H}^*) \longrightarrow H^1(F^* \otimes \mathcal{J}_H) \longrightarrow H^1(F^*)$$

we get that the kernel of $\phi : H^1(F^* \otimes \mathcal{J}_H) \to H^1(F^*)$ has dimension at least $a$. This kernel, regarded as a subspace of $S^* \otimes H^0(\mathcal{J}_H(1))$ (see Lemma 2.1(i)), is necessarily of the form $A \otimes H^0(\mathcal{J}_H(1))$, because $H^0(\mathcal{J}_H(1))$ has dimension 1. Therefore, $(A, H)$ is an $(a, 1)$-jumping pair and $H$ is an $(a, 1)$-jumping hyperplane.   □

REMARK 2.3.   Since Steiner bundles of rank $n$ are reduced (see Remark 1.4), part (iv) of Lemma 2.2 says that a jumping hyperplane $H$ is characterized by the condition $H^0(F_{|H}^*) \neq 0$. This is why in [**1**, **9**] is used the name 'unstable hyperplane', although in our general context we preferred the word 'jumping'. Observe that part (iii) implies that, if $\Lambda$ is an $(a, b)$-jumping subspace of $F$, then $h^0(F_{|\Lambda}^*) \geqslant h^0(F^*) + ab$. However, the converse is not true, and the proof of (iv) does not work if $b > 1$, since an $ab$-dimensional kernel of $H^1(F^* \otimes \mathcal{J}_\Lambda) \to H^1(F^*)$ is not necessarily of the form $A \otimes H^0(\mathcal{J}_\Lambda(1))$. However, one could characterize $(a, b)$-jumping pairs $(A, \Lambda)$ by the property that, for any hyperplane $H \supset \Lambda$, the pair $(A, H)$ is an $(a, 1)$-jumping pair or, similarly, that for any hyperplane $H \supset \Lambda$ and any line $A' \subset A$ the pair $(A', H)$ is a jumping pair.

The reader should notice however that, when $b = n - 1$, our notion of jumping hyperplane does not coincide with the standard notion of jumping line of a vector bundle in the projective space, even if $n = 2$ (that is, $b = 1$). For instance, the Steiner bundle $F = S^2(T_{\mathbb{P}^2}(-1))$ of Example 1.10 is uniform, and even homogeneous, so that it has no jumping lines (in the standard sense), while any line $L \subset \mathbb{P}^2$ is a jumping hyperplane (in our sense) because $F_{|L}$ has always a trivial summand.

We can give a geometric construction of the sets of the $(a, b)$-jumping subspaces and pairs, which endows them with a natural structure of algebraic sets (when $a = b = 1$, this is the natural generalization of the construction given in [**1**, §3] for Steiner bundles of rank $n$). This also allows to show that, when these sets satisfy certain conditions of linear normality, the answer to Question 0.1 is positive.

LEMMA 2.4. *Let $F$ be a Steiner bundle over $\mathbb{P}^n$ and let $T_0^* \subset S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ be the image of $\varphi$. Consider the natural generalized Segre embedding*

$$\nu : G(a, S^*) \times G(b, H^0(\mathcal{O}_{\mathbb{P}^n}(1))) \longrightarrow G(ab, S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)))$$

*(given by the tensor product of subspaces) and identify $G(b, H^0(\mathcal{O}_{\mathbb{P}^n}(1)))$ with the Grassmann variety of subspaces of codimension $b$ in $\mathbb{P}^n$. Then we have the following.*

(i) *The set $\tilde{J}_{a,b}(F)$ of jumping pairs of $F$ is the intersection of the image of $\nu$ with the subset $G(ab, T_0^*) \subset G(ab, S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)))$.*

(ii) *If $\pi_1, \pi_2$ are the respective projections from $\tilde{J}_{a,b}(F)$ to $G(a, S^*)$ and $G(b, H^0(\mathcal{O}_{\mathbb{P}^n}(1)))$, then $\Sigma_{a,b}(F) = \pi_1(\tilde{J}_{a,b}(F))$ and $J_{a,b}(F) = \pi_2(\tilde{J}_{a,b}(F))$.*

(iii) *Let $\mathcal{A}, \mathcal{B}, \mathcal{Q}$ be the universal quotient bundles of respective ranks $a, b, ab$ of $G(a, S^*)$, $G(b, H^0(\mathcal{O}_{\mathbb{P}^n}(1)))$ and $G(ab, T_0^*)$. Assume that the natural maps*

$$\alpha : H^0(G(a, S^*), \mathcal{A}) \longrightarrow H^0(\tilde{J}_{a,b}(F), \pi_1^* \mathcal{A}),$$
$$\beta : H^0(G(b, H^0(\mathcal{O}_{\mathbb{P}^n}(1))), \mathcal{B}) \longrightarrow H^0(\tilde{J}_{a,b}(F), \pi_2^* \mathcal{B}),$$
$$\gamma : H^0(G(ab, T_0'^*), \mathcal{Q}) \longrightarrow H^0(\tilde{J}_{a,b}(F), \mathcal{Q}_{|\tilde{J}_{a,b}(F)})$$

*are isomorphisms. Then the reduced summand $F_0$ of $F$ is the Schwarzenberger bundle of the triplet $(\tilde{J}_{a,b}(F), \pi_1^* \mathcal{A}, \pi_2^* \mathcal{B})$.*

*Proof.* Part (i) comes immediately from Lemma 2.2(ii), while part (ii) comes from the definition of $\Sigma_{a,b}(F)$ and $J_{a,b}(F)$.

For part (iii), observe that there is a commutative diagram

$$\begin{array}{ccc}
S \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))^* & \longrightarrow & T_0' \\
\downarrow & & \downarrow \\
H^0(\tilde{J}_{a,b}(F), \pi_1^* \mathcal{A}) \otimes H^0(\tilde{J}_{a,b}(F), \pi_2^* \mathcal{B}) & \longrightarrow & H^0(\tilde{J}_{a,b}(F), \pi_1^* \mathcal{A} \otimes \pi_2^* \mathcal{B})
\end{array}$$

in which we have the following.

(i) The top map is the dual of the inclusion $T_0'^* \to S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, which is naturally identified with the map

$$H^0(G(a, S^*), \mathcal{A}) \otimes H^0(G(b, H^0(\mathcal{O}_{\mathbb{P}^n}(1)), \mathcal{B}) \longrightarrow H^0(G(ab, T_0'^*), \mathcal{Q})$$

consisting of the restriction from $G(ab, S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)))$ to $G(ab, T_0'^*)$ of the sections of the universal quotient bundle of rank $ab$.

(ii) The vertical maps are, with the above identifications, $\alpha \otimes \beta$ and $\gamma$, so that they are isomorphisms by hypothesis.

(iii) The bottom map is the multiplication map whose dual, by Example 1.5, defines (in the sense of Lemma 1.2) the Schwarzenberger bundle of the triplet $(\tilde{J}_{a,b}(F), \pi_1^*\mathcal{A}, \pi_2^*\mathcal{B})$.

Since the dual of the top map is the one defining (in the sense of Lemma 1.2), the bundle $F_0$, part (iii) follows from the vertical isomorphisms.                                                             □

EXAMPLE 2.5.   We illustrate the above situation in the case $a = b = 1$, the one on which we will concentrate in this paper. In this case, $\tilde{J}(F)$ is the intersection of the Segre variety $\mathbb{P}(S) \times \mathbb{P}^{n*}$ with the projective space $\mathbb{P}(T_0)$. The conditions of Lemma 2.4(iii) are the linear normality and nondegeneracy, respectively, of $\tilde{J}(F)$ in $\mathbb{P}(T_0)$, of $\Sigma(F)$ in $\mathbb{P}(S)$, and of $J(F)$ in $\mathbb{P}^{n*}$. Using the standard properties of the classical Segre embedding, we will have the following properties that we will use frequently:

   (i) The set $\tilde{J}(F)$ is cut out by quadrics.
   (ii) The fibres of $\pi_1, \pi_2$ are linear subspaces of $\mathbb{P}(T_0)$.
   (iii) Any linear subspace of $\tilde{J}(F)$ is contained in a fibre of $\pi_1$ or $\pi_2$.

Depending on the context, we will regard $\tilde{J}(F)$ as a subvariety of the projective space $\mathbb{P}(T_0)$ or as a subvariety of the product $\mathbb{P}(S) \times \mathbb{P}^{n*}$. It will be useful to observe that the relation among these two points of view is that the hyperplane section of $\tilde{J}(F)$ as a subvariety of $\mathbb{P}(T_0)$ is $\pi_1^*\mathcal{O}_{\mathbb{P}(S)}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^{n*}}(1)$, where $\pi_1, \pi_2$ are the projections to $\mathbb{P}(S)$ and $\mathbb{P}^{n*}$.

REMARK 2.6.   Observe that, in general, one should not expect the hypothesis of Lemma 2.4(iii) to hold. This is because the condition (ii) in Lemma 1.2 is open in the set of linear maps $\varphi : T^* \to S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$. Hence a general $\varphi$ will produce a Steiner bundle, which will also be reduced. Since $G(a, S^*) \times G(b, H^0(\mathcal{O}_{\mathbb{P}^n}(1)))$ tends to have a big codimension in $G(ab, S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)))$, one should expect its intersection with a general $G(ab, T^*)$ to be very small, and in general empty. Therefore, for arbitrary big values of $s, t, a, b$, the set $\tilde{J}_{a,b}(F)$ is expected to be empty, that is, a general Steiner bundle will not have jumping $(a, b)$-subspaces.

For example, if $s = 3, t = n + 4$, a general $(3, n + 4)$-Steiner bundle on $\mathbb{P}^n$ does not have jumping hyperplanes when $n \geqslant 4$, since the Segre variety $\mathbb{P}^2 \times \mathbb{P}^n$ has codimension $2n$ in $\mathbb{P}^{3n+2}$, so its intersection with a general linear space of dimension $n + 3$ is empty. This also shows that, for $n = 2$, the set of jumping pairs of a general $F$ is a curve in $\mathbb{P}^2$, so that $F$ cannot be the Schwarzenberger bundle of the triplet $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(1))$ (see Example 1.10). However, we will see in Theorem 2.8(iv) that, when $s = 2$, the expected dimension of the set of jumping pairs is 'the right one'.

Our goal now is to see that the hypothesis of Lemma 2.4(iii) holds if $F$ has 'many' jumping pairs. The first thing we will need to do is to understand how big the dimension of $\tilde{J}(F)$ can be. By Example 2.5, we need to study how the Segre variety can intersect linear subspaces of given dimension. To do so, we need a technical result of linear algebra (in which it is crucial that the ground field is algebraically closed), which we state as a separate lemma. Even if we are going to use it only for $a = b = 1$, we include the general statement, since the general proof does not add any difficulty and since we hope that it could be useful in a future work.

LEMMA 2.7.   *Let $U, V$ be two vector spaces of respective dimensions $r, s$ over the algebraically closed field $k$. Fix nonzero subspaces $B \subset U$ of codimension $b < r$ and $A \subset V$ of dimension $a < s$. Let $W$ be a $t$-dimensional linear space of $\mathrm{Hom}(U, V)$ such that for any $u \in U$ and any $v \in V$ there exists $f \in W$ such that $f(u) = v$. Then*

$$\dim\{f \in W \mid f(B) \subset A\} \leqslant t - r - s + a + b + 1.$$

*Proof.* We take any basis $v_1, \ldots, v_s$ of $V$ such that $v_1, \ldots, v_a \in A$ and pick also any nonzero vector $u_1 \in B$. By assumption, there exist linear maps $g_{a+1}, \ldots, g_s$ in $W$ such that $g_i(u_1) = v_i$ for $i = a+1, \ldots, s$.

Let us construct next, for $i = 2, \ldots, r-b$, vectors $u_1, \ldots, u_i \in B$ and maps $h_2, \ldots, h_i \in W$ such that

$$h_i(u_i) \notin \langle v_1, \ldots, v_a, g_{a+1}(u_i), \ldots, g_s(u_i), h_2(u_i), \ldots, h_{i-1}(u_i) \rangle \quad \text{for } i = 2, \ldots, r-b.$$

We do it by iteration, so we can assume that we have already constructed $u_1, \ldots, u_{i-1}$ and $h_2, \ldots, h_{i-1}$. Take any $u_i' \in B \setminus \langle u_1, \ldots, u_{i-1} \rangle$ (we can do so because $i-1 \leqslant r-b-1 < \dim B$). For any $\lambda_1, \ldots, \lambda_i$, consider the vectors

$$v_1, \ldots, v_a, g_{a+1}(\lambda_1 u_1 + \ldots + \lambda_{i-1} u_{i-1} + \lambda_i u_i'), \ldots, g_s(\lambda_1 u_1 + \ldots + \lambda_{i-1} u_{i-1} + \lambda_i u_i'),$$
$$h_2(\lambda_1 u_1 + \ldots + \lambda_{i-1} u_{i-1} + \lambda_i u_i'), \ldots, h_{i-1}(\lambda_1 u_1 + \ldots + \lambda_{i-1} u_{i-1} + \lambda_i u_i')$$

and the $(s+i-2) \times s$ matrix given by their coordinates with respect to $v_1 \ldots, v_s$. This matrix will have no maximal rank if and only if the $(s-a+i-2) \times (s-a)$ submatrix obtained by removing the first $a$ rows and columns has no maximal rank. The assumption $s > a$ implies that this submatrix is not vacuous, and since its entries are linear forms in $\lambda_1, \ldots, \lambda_i$ and the ground field is algebraically closed, there exists some nonzero solution $\lambda_1, \ldots, \lambda_i$ for which the submatrix has not maximal rank. We take $u_i = \lambda_1 u_1 + \ldots + \lambda_{i-1} u_{i-1} + \lambda_i u_i'$ for some nonzero solution as above. Hence there exists $v \in V \setminus \langle v_1, \ldots, v_a, g_{a+1}(u_i), \ldots, g_s(u_i), h_2(u_i), \ldots, h_{i-1}(u_i) \rangle$. We thus take $h_i \in W$ such that $h_i(u_i) = v$, which completes the iteration process.

Assume that we know that $g_{a+1}, \ldots, g_s, h_2, \ldots, h_{r-b} \in W$ are linearly independent modulo $\{f \in W \mid f(B) \subset A\}$. This would imply that, inside the vector space $W$, the subspace $\{f \in W \mid f(B) \subset A\}$ has zero intersection with the $(r+s-a-b-1)$-dimensional subspace generated by $g_{a+1}, \ldots, g_s, h_2, \ldots, h_{r-b}$. We would get then the wanted inequality.

We are thus left to prove that $g_{a+1}, \ldots, g_s, h_2, \ldots, h_{r-b} \in W$ are linearly independent modulo $\{f \in W \mid f(B) \subset A\}$. Assume that we have some linear combination

$$f := \mu_{a+1} g_{a+1} + \ldots + \mu_s g_s + \nu_2 h_2 + \ldots + \nu_{r-b} h_{r-b}$$

such that $f(B) \subset A = \langle v_1, \ldots, v_a \rangle$. Applying both terms to $u_{r-b}$, we get

$$\nu_{r-b} h_{r-b}(u_{r-b}) \in \langle v_1, \ldots, v_a, g_{a+1}(u_{r-b}), \ldots, g_s(u_{r-b}), h_2(u_{r-b}), \ldots, h_{r-b-1}(u_{r-b}) \rangle,$$

which implies $\nu_{r-b} = 0$, by our choice of $u_{r-b}$. Knowing this vanishing, we consider now $f(u_{r-b-1})$ and get $\nu_{r-b-1} = 0$ in the same way, and iterating we get $\nu_2 = \ldots = \nu_{r-b} = 0$. We thus have $f(u_1) = \mu_{a+1} v_{a+1} + \ldots + \mu_s v_s$, which implies now $\mu_{a+1}, \ldots, \mu_s = 0$ since $f(u_1) \in \langle v_1, \ldots, v_a \rangle$. $\qquad\square$

We can now give, for $a = b = 1$, an upper bound for the dimension of the set of jumping pairs. Since Lemma 2.4 gives $J(F) = \pi_2(\tilde{J}(F))$, the same bound will hold for the dimension of the set of jumping hyperplanes. Observe that our bound is sharp, because it is achieved in the cases of Examples 1.7–1.10 (since at least the points of $X$ provide jumping pairs).

THEOREM 2.8. *Let $F$ be an $(s, t)$-Steiner bundle on $\mathbb{P}^n$ with $s \geqslant 2$. Then we have the following.*

(i) *The embedded Zariski tangent space at any point of $\tilde{J}(F)$ has dimension at most $t - n - s + 1$; in particular, $\dim \tilde{J}(F) \leqslant t - n - s + 1$.*

(ii) *If $\tilde{J} \subset \mathbb{P}(S) \times \mathbb{P}^{n*}$ is a component of $\tilde{J}(F)$ such that its projection to $\mathbb{P}(S)$ or $\mathbb{P}^{n*}$ is constant, then $\dim \tilde{J} < t - n - s + 1$.*

(iii) *If $\tilde{J}(F)$ has dimension $t - n - s + 1$, then $F$ is reduced and $\tilde{J}(F)$ is smooth at the points of any of its irreducible components of maximal dimension.*

(iv) *If $s = 2$ and $F$ is reduced, then $\tilde{J}(F)$ is a rational normal scroll of dimension $t - n - 1$* (*and degree $n + 1$*) *and $F$ is the Schwarzenberger bundle of Example* 1.9.

(v) *If $n = 1$ and $F$ is reduced, then $\tilde{J}(F)$ is a rational normal scroll of dimension $t - s$ (and degree $s$) and $F$ is the Schwarzenberger bundle of Example* 1.8.

*Proof.* To prove (i), we identify $\mathbb{P}(S \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*)$ with the set of nonzero linear maps (up to multiplication by a constant) $H^0(\mathcal{O}_{\mathbb{P}^n}(1))^* \to S^*$. Then the Segre variety corresponds to maps of rank 1, while $\mathbb{P}(T_0)$ corresponds to the subspace $T_0^* \subset \mathrm{Hom}(H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*, S^*)$ of Lemma 1.3. Fix any point $(\alpha, H) \in \tilde{J}(F) \subset \mathbb{P}(S) \times \mathbb{P}^{n*}$. As a point in $\mathbb{P}(S \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*)$, it is represented by a linear map $H^0(\mathcal{O}_{\mathbb{P}^n}(1))^* \to S^*$ whose kernel is a hyperplane $\vec{H} \subset H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*$ defining $H$ and whose image is a line $A \subset S^*$ representing $\alpha$. The embedded tangent space to the Segre variety at $(\alpha, H)$ corresponds then to the subspace of linear maps $f : H^0(\mathcal{O}_{\mathbb{P}^n}(1))^* \to S^*$ such that $f(\vec{H}) \subset A$ (see, for instance, [**4**, Example 14.16]). Since $\tilde{J}(F)$ is the intersection of the Segre variety with $\mathbb{P}(T_0)$, it follows that its embedded tangent space at $(\alpha, H)$ corresponds to the subspace of linear maps $f \in T_0^*$ such that $f(\vec{H}) \subset A$. By Lemma 2.7 (whose hypotheses are satisfied by Lemma 1.2), this subspace has dimension at most $t_0 - (n + 1) - s + 3$, where $t_0 = \dim T_0$. Since $t_0 \leqslant t$, it follows that the dimension of the embedded Zariski tangent space of $\tilde{J}(F)$ at $(\alpha, H)$ is at most $t - n - s + 1$, which completes the proof of (i).

In order to prove (ii), assume first that the image of $\tilde{J}$ in $\mathbb{P}(S)$ is a point corresponding to a line $A \subset S^*$. Then the embedded tangent space at any point of $\tilde{J}$ is contained in the subspace corresponding to the linear maps $f \in T_0^*$ such that $f(H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*) \subset A$. By Lemma 2.7 (taking $B = H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*$), we get, arguing as in (i), that the embedded tangent space would have dimension at most $t - n - s$, as wanted. If instead the image of $\tilde{J}$ in $\mathbb{P}^{n*}$ is an element corresponding to a hyperplane $B \subset H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*$, we proceed in the same way: now the embedded tangent space of $\tilde{J}$ is contained in the subspace corresponding to the linear maps $f \in T_0^*$ such that $f(B) = 0$, and we use Lemma 2.7 taking $A = 0$.

To prove (iii), assume that we have $\dim \tilde{J}(F) = t - n - s + 1$. Hence in the proof of (i) all inequalities are equalities. In particular $t_0 = t$, so that $F$ is reduced. On the other hand, for any component of $\tilde{J}(F)$ of dimension $t - n - s + 1$, the dimension of its embedded tangent space at any point cannot exceed $t - n - s + 1$, by (i), so that all the points of that component are smooth.

Assume now $s = 2$ in order to prove (iv). In this case $\mathbb{P}(S) \times \mathbb{P}^{n*}$ has codimension $n$ in $\mathbb{P}(S \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))^*)$, so that its intersection with $\mathbb{P}(T)$ has dimension at least $t - 1 - n$. By (iii), it follows that $\tilde{J}(F)$ is a smooth complete intersection of $\mathbb{P}(S) \times \mathbb{P}^{n*}$ and $\mathbb{P}(T)$, that is, a smooth rational normal scroll $\tilde{J}(F) \subset \mathbb{P}(T)$ of dimension $t - n - 1$, so that we can make the identification $T = H^0(\mathcal{O}_{\tilde{J}(F)}(h))$, where $h$ is the hyperplane section class of the scroll. It also follows from (ii) that the projection $\pi_1 : \tilde{J}(F) \to \mathbb{P}(S) = \mathbb{P}^1$ is not constant, hence it is surjective. Therefore all the fibres of $\pi_1$ (which are linear spaces, by Example 2.5(ii)) have dimension $t - n - 2$, so that $\pi_1$ gives the scroll structure on $\tilde{J}(F)$. We can thus identify $S = H^0(\mathcal{O}_{\tilde{J}(F)}(f))$, where $f$ is the class of a fibre of the scroll and, as pointed out in Example 2.5, the map from $\tilde{J}(F)$ to $\mathbb{P}^{n*}$ is given by $\mathcal{O}_{\tilde{J}(F)}(h - f)$. In order to complete the proof of (iv) we need to show, by Lemma 2.4(iii), that we can identify $H^0(\mathcal{O}_{\mathbb{P}^n}(1))^* = H^0(\mathcal{O}_{\tilde{J}(F)}(h - f))$. This identification comes from the fact that the restriction map $H^0(\mathcal{O}_{\mathbb{P}(S) \times \mathbb{P}^{n*}}(0, 1)) \to H^0(\mathcal{O}_{\tilde{J}(F)}(h - f))$ is an isomorphism because $\tilde{J}(F)$ is the complete intersection of $\mathbb{P}(S) \times \mathbb{P}^{n*}$ and a linear space.

Finally, (v) was proved in Example 1.8 (observe that a Steiner bundle on $\mathbb{P}^1$ is reduced if and only if it is ample), although the same proof as in (iv) holds. $\square$

REMARK 2.9. Observe that part (iv) of Theorem 2.8 is giving more information about Example 1.9. Indeed our proof shows that we have $X = \tilde{J}(F)$, even with the scheme structure

of $\tilde{J}(F)$ as intersection of the Segre variety and a linear space, and shows in particular that any jumping hyperplane of $F$ is coming from a point of $X$. Hence, for the Schwarzenberger bundles of Example 1.9, we get a positive answer to Question 0.2 (the same holds for Example 1.8). Incidentally, observe that, in this example, the set of jumping hyperplanes has not always maximal dimension $t - n - 1$. This is because $J(F)$ is the image of the rational normal scroll $X$ via $\mathcal{O}_X(h - f)$, which drops dimension if (and only if) $X$ is the Segre variety $\mathbb{P}^1 \times \mathbb{P}^n$ (which is equivalent to saying $t = 2n + 2$), in which $\mathcal{O}_X(h - f)$ induces the projection onto $\mathbb{P}^n$. In particular, in this last case, all the hyperplanes are jumping hyperplanes. We also remark that, as pointed out by the referee, parts (iv) and (v) of Theorem 2.8 are well known in the language of Kronecker canonical forms, although we kept our proofs for the sake of uniformity of the exposition.

Observe also that, in general, the answer to Question 0.2 can be negative. For example, if $X$ is an elliptic curve and $L, M$ are line bundles on $X$ of respective degrees 2 and $n + 1$, the Schwarzenberger bundle of the triplet $(X, L, M)$ is a $(2, n + 3)$-Steiner bundle $F$. However, Theorem 2.8(iv) implies that $\tilde{J}(F)$ and $J(F)$ are rational normal scrolls of dimension 2 instead of just the original elliptic curve $X$ (it can be seen that these scrolls consist of the union of the lines spanned by the pairs of points of $X$ given by the divisors in the linear system defined by $L$).

## 3. Steiner bundles with jumping locus of maximal dimension

In this section we will characterize $(s, t)$-Steiner bundles for which $\tilde{J}(F)$ has the maximal dimension $t - n - s + 1$, showing that they are exactly Examples 1.7–1.10 (hence we give a positive answer to Question 0.1 in this case). When the maximal dimension is 1 (that is, when $t = n + s$ or, equivalently, $F$ has rank $n$), we recover the known result that Steiner bundles of rank $n$ with a curve of jumping hyperplanes are precisely the classical Schwarzenberger bundles (see [**9**]).

The main idea, borrowed from the case of rank $n$, will be to produce, from a given $(s, t)$-Steiner bundle, an $(s - 1, t - 1)$-Steiner bundle (thus with the same rank as $F$) with essentially the same jumping hyperplanes. Then, after an iteration, we will eventually arrive at a Steiner bundle with $s = 2$ to which we can apply Theorem 2.8(iv). Analogously, we will produce an $(s, t - 1)$-Steiner bundle on a (jumping) hyperplane, and eventually arrive at a Steiner bundle on $\mathbb{P}^1$ to which we can apply Theorem 2.8(v) (we will omit the details of this second iteration, stating the results we will need in Remark 3.5).

The starting point is the following (see [**9**, Proposition 2.1] for the case of rank $n$).

PROPOSITION 3.1. *Let $F$ be a reduced $(s, t)$-Steiner bundle on $\mathbb{P}^n$, and let $\pi_1, \pi_2$ denote the two projections from $\tilde{J}(F) \subset \mathbb{P}(S) \times \mathbb{P}^{n*}$. Let $(\alpha, H)$ be a jumping pair of $F$, let $i : S' \subset S$ and $j : T' \subset T$ be the hyperplane inclusions corresponding, respectively, to $\alpha \in \mathbb{P}(S)$ and $(\alpha, H) \in \mathbb{P}(T)$. If $F'$ is the kernel of the natural composition $F \to F_{|H} \to \mathcal{O}_H$ defined by $(\alpha, H)$ (see Lemma 2.2(iii)) then:*

(i) *$F'$ is an $(s - 1, t - 1)$-Steiner bundle $F'$ having a resolution*

$$0 \longrightarrow S' \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow T' \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow F' \longrightarrow 0;$$

(ii) *the linear map $\varphi'$ defining $F'$ (see Lemma 1.2) fits in a commutative diagram*

$$\begin{array}{ccc} T^* & \xrightarrow{\varphi} & S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \\ \downarrow j^* & & \downarrow i^* \otimes \mathrm{id} \\ T'^* & \xrightarrow{\varphi'} & S'^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1)); \end{array}$$

(iii) *$J(F) \subset J(F') \cup \pi_2 \pi_1^{-1}(\alpha)$.*

*Proof.* We have the following commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S' \otimes \mathcal{O}_{\mathbb{P}^n}(-1) & \longrightarrow & T' \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & F' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S \otimes \mathcal{O}_{\mathbb{P}^n}(-1) & \longrightarrow & T \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & F & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(-1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & \mathcal{O}_H & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

where the first column is defined by the quotient of $S$ corresponding to $\alpha$, the second column is defined by the quotient of $T$ corresponding to $(\alpha, H)$, and the first row is defined as a kernel. This proves (i).

Taking duals, we get another commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F^* & \longrightarrow & T^* \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & S^* \otimes \mathcal{O}_{\mathbb{P}^n}(1) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F'^* & \longrightarrow & T'^* \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & S'^* \otimes \mathcal{O}_{\mathbb{P}^n}(1) & \longrightarrow & 0
\end{array}
$$

which, taking cohomology, produces (ii).

To prove (iii), consider any jumping hyperplane $H_1$ of $F$ and assume it is not in $\pi_2\pi_1^{-1}(\alpha)$, so that it comes from a jumping pair $(\alpha_1, H_1)$ with $\alpha_1 \neq \alpha$. This jumping pair is represented by a nonzero tensor $v_1 \otimes h_1 \in S^* \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ in the image of $\varphi$ (where $h_1$ is an equation of $H_1$). Since $\alpha_1 \neq \alpha$, it follows that $i^*(v_1) \otimes h_1$ is nonzero, and it is also in the image of $\varphi'$, by (ii). This implies that $([i^*(v_1)], H_1)$ is a jumping pair of $F'$, so that $H_1$ is a jumping hyperplane of $F'$, as wanted. □

REMARK 3.2. The idea now is that, when performing the iteration process, part (iii) of Proposition 3.1 should provide enough information to keep track of the set of jumping pairs until we arrive at a Steiner bundle with $s = 2$. There are two difficulties to do so. First of all, some bundle in the iteration process could be nonreduced, although we could deal with this taking its reduced summand and using Lemma 2.2(i). The main difficulty is however that Proposition 3.1(iii) does not relate $J(F)$ and $J(F')$ if $J(F)$ is contained in some $\pi_2\pi_1^{-1}(\alpha)$. Of course this behaviour seems very unlikely (for instance, it does not hold if $\dim \tilde{J}(F) = t - n - s + 1$, as Theorem 2.8(ii) guarantees), and we could impose that it does not hold for our original $F$, but still it could hold for some other Steiner bundle in the iteration process.

In the case of Steiner bundles of rank $n$ (the one studied in [**9**]), which are always reduced, this last difficulty can be avoided as follows. Any Steiner bundle $F'$ in the process has rank $n$, so that from Theorem 2.8(i) its set of jumping hyperplanes has dimension at most 1. Therefore, if the projection $\pi_1' : \tilde{J}(F') \to \mathbb{P}(S')$ were constant, its fibre (which is a linear space, by Example 2.5(ii)) would be either a point or a line. It cannot be a line by Theorem 2.8(ii), so that necessarily $F'$ would have only one jumping hyperplane. This is the key underlying idea in [**9**] that allows even to limit the number of jumping hyperplanes when there are finitely many.

The key to deal with the first difficulty of Remark 3.2 is the following (in which we also pay attention to jumping pairs instead of just jumping hyperplanes).

PROPOSITION 3.3. *In the situation of Proposition 3.1, set $T_0'^* := \mathrm{Im}\,\varphi'$ and let $F' = F_0' \oplus (T'/T_0') \otimes \mathcal{O}_{\mathbb{P}^n}$ be the decomposition of Lemma 1.3. Then we have the following.*

(i) *The projection from the linear subspace $\pi_1^{-1}(\alpha) \subset \mathbb{P}(T)$ is the map $\mathrm{pr}_{(\alpha,H)} : \mathbb{P}(T) \to \mathbb{P}(T_0')$ induced by the composition $T_0' \xrightarrow{j^*} T' \xrightarrow{\varphi'} T$. In particular, $\dim T_0' = t - 1 - \dim \pi_1^{-1}(\alpha)$.*

(ii) *If $\mathrm{pr}_\alpha : \mathbb{P}(S) \to \mathbb{P}(S')$ denotes the projection from $\alpha$, for any $(\alpha_1, H_1)) \in \tilde{J}(F)$ with $\alpha_1 \neq \alpha$, we have the equality*

$$\mathrm{pr}_{(\alpha,H)}(\alpha_1, H_1) = (\mathrm{pr}_\alpha(\alpha_1), H_1)$$

*and this is a jumping pair of $F'$ and $F_0'$.*

(iii) *The variety $\tilde{J}(F_0')$ contains the image under $\mathrm{pr}_{(\alpha,H)}$ of any component of $\tilde{J}(F) \subset \mathbb{P}(T)$ not contained in $\pi_1^{-1}(\alpha)$.*

(iv) *The variety $\Sigma(F_0')$ contains the image under $\mathrm{pr}_\alpha$ of any component of $\Sigma(F) \subset \mathbb{P}(S)$ different from $\{\alpha\}$.*

*Proof.* It follows readily from the commutative diagram of Proposition 3.1(ii). For example, part (i) comes from the fact that the subspace of $T^*$ corresponding to $\pi_1^{-1}(\alpha)$ is the kernel of $(i^* \otimes id) \circ \varphi = \varphi' \circ j^*$. Part (ii) is now the interpretation of the diagram of Proposition 3.1(ii) (recall that $F'$ and $F_0'$ has the same jumping pairs, by Lemma 2.2(i)). Finally, parts (iii) and (iv) are proved from (ii) (in fact, it is the same proof as the one of Proposition 3.1(iii)). $\square$

The next proposition shows that, for Steiner bundles of arbitrary rank, the second difficulty of Remark 3.2 can be overcome with the same ideas as in the case of rank $n$ if we assume that the set of jumping pairs has the maximal dimension allowed by Theorem 2.8(i) (observe that, in this case, the bundle is necessarily reduced, by Theorem 2.8(iii)).

PROPOSITION 3.4. *Let $F$ be an $(s,t)$-Steiner bundle on $\mathbb{P}^n$ with $s \geqslant 2$ and such that $\tilde{J}(F)$ has dimension $t - n - s + 1$. Let $\tilde{J}_0$ be a component of $\tilde{J}(F)$ of maximal dimension and fix $(\alpha, H) \in \tilde{J}_0$. Then, if $F'$ is the Steiner bundle constructed in Proposition 3.1 and $F_0'$ is its reduced part, the following hold.*

(i) *The image of both $\tilde{J}_0$ and $\tilde{J}(F)$ under the projection $\mathrm{pr}_{(\alpha,H)}$ from $\pi_1^{-1}(\alpha)$ has dimension $t - n - s + 1 - \dim \pi_1^{-1}(\alpha)$.*

(ii) *The variety $\tilde{J}(F_0')$ has dimension $t - n - s + 1 - \dim \pi_1^{-1}(\alpha)$.*

(iii) *If $\tilde{J}(F_0')$ is irreducible, then:*
   (a) *$\tilde{J}(F_0')$ is the image of $\tilde{J}(F)$ under the projection $\mathrm{pr}_{(\alpha,H)}$ from $\pi_1^{-1}(\alpha)$;*
   (b) *$\tilde{J}(F)$ is irreducible;*
   (c) *$J(F) = J(F_0')$;*
   (d) *$\Sigma(F_0')$ is the image of $\Sigma(F)$ under the inner projection $\mathrm{pr}_\alpha$ from $\alpha$.*

*Proof.* Since, by Theorem 2.8(i), $\tilde{J}(F_0')$ has dimension at most $\dim T_0' - n - (s-1) + 1$ and, by Proposition 3.3(i), $\dim T_0' = t - 1 - \dim \pi_1^{-1}(\alpha)$, part (i) will follow if we prove that the image of $\tilde{J}_0$ under $\mathrm{pr}_{(\alpha,H)}$ has dimension at least $t - n - s + 1 - \dim \pi_1^{-1}(\alpha)$. Assume by contradiction that $\tilde{J}_0$ drops dimension by $\dim \pi_1^{-1}(\alpha) + 1$ when projecting from $\pi_1^{-1}(\alpha)$. This means that $\tilde{J}_0$ is a cone with vertex $\pi_1^{-1}(\alpha)$. Since any line in the cone is contained in a fibre of $\pi_1$ or $\pi_2$ (Example 2.5(iii)), it follows that $\tilde{J}_0$ is contained in $\pi_1^{-1}(\alpha) \cup \pi_2^{-1}(H)$. But $\tilde{J}_0$ is irreducible, so that it is contained in $\pi_1^{-1}(\alpha)$ or $\pi_2^{-1}(H)$, which contradicts Theorem 2.8(ii).

To prove (ii), we have, on the one hand, that Proposition 3.3(iii) implies that $\tilde{J}(F')$ contains the image of $\tilde{J}_0$ under $\mathrm{pr}_{(\alpha,H)}$, which has dimension $t - n - s + 1 - \dim \pi_1^{-1}(\alpha)$, by (i). On the other hand, Theorem 2.8(i) implies $\dim \tilde{J}(F') \leqslant t - n - s + 1 - \dim \pi_1^{-1}(\alpha)$, so that (ii) follows.

To prove (iii), observe first that $\tilde{J}(F)$ cannot have any component contained in $\pi_1^{-1}(\alpha)$. Indeed $\pi_1^{-1}(\alpha)$ is contained in $\tilde{J}_0$, since otherwise it would be contained in another component

of $\tilde{J}(F)$. But then such a component would meet $\tilde{J}_0$ at least at the point $(\alpha, H)$, implying that $(\alpha, H)$ is a singular point of $\tilde{J}(F)$, contradicting Theorem 2.8(iii).

I claim now that $\tilde{J}(F')$ coincides with the image of both $\tilde{J}_0$ and $\tilde{J}(F)$ under $\mathrm{pr}_{(\alpha,H)}$. Indeed, both images are contained in $\tilde{J}(F')$ by Proposition 3.3(iii) (and the above observation), and on the other hand they have dimension $t - n - s + 1 - \dim \pi_1^{-1}(\alpha)$, by (i). Since, by (ii), $\tilde{J}(F')$ has also dimension $t - n - s + 1 - \dim \pi_1^{-1}(\alpha)$, its irreducibility proves the claim, and also part (a).

To prove part (b), assume for contradiction that $\tilde{J}(F)$ has another component $\tilde{J}_1$ different from $\tilde{J}_0$, and fix any point $(\alpha_1, H_1) \in \tilde{J}_1 \setminus \tilde{J}_0$. By our previous claim, the image of $(\alpha_1, H_1)$ under $\mathrm{pr}_{(\alpha,H)}$ is also in the image of $\tilde{J}_0$. In particular, there is a line $\Delta$ trisecant to $\tilde{J}(F)$, passing through $(\alpha_1, H_1)$ and meeting $\pi_1^{-1}(\alpha)$. Since $\tilde{J}(F)$ is cut out by quadrics (Example 2.5(i)), it follows that $\Delta$ is contained in $\tilde{J}(F)$. But $\Delta \not\subset \tilde{J}_0$, so that there is another component of $\tilde{J}(F)$ containing $\Delta$. Therefore $\tilde{J}_0$ meets that component at the point $(\alpha, H)$, so that $(\alpha, H)$ is a singular point of $\tilde{J}(F)$ that is in $\tilde{J}_0$. This contradicts once more Theorem 2.8(iii), hence (b) holds.

We prove part (c) by showing the double inclusion. Observe first that the irreducibility of $\tilde{J}(F)$ implies the irreducibility of $J(F)$. Thus, Proposition 3.1(iii) implies, together with Theorem 2.8(ii), that $J(F)$ is contained in $J(F')$, which is $J(F'_0)$ by Lemma 2.2(i), so that we are left to prove the other inclusion. Since $\mathrm{pr}_{(\alpha,H)}(\tilde{J}(F) \setminus \pi_1^{-1}(\alpha))$ is dense in $\tilde{J}(F'_0)$, also $\pi'_1(\mathrm{pr}_{(\alpha,H)}(\tilde{J}(F) \setminus \pi_1^{-1}(\alpha)))$ is dense in $J(F'_0)$, so it is enough to prove that any element of it is also in $J(F)$. We thus take $H' \in J(F'_0)$ for which there exists $\alpha' \in \mathbb{P}(S')$ such that $(\alpha', H') = \mathrm{pr}_{(\alpha,H)}(\alpha_1, H_1)$ for some $(\alpha_1, H_1) \in \tilde{J}(F)$ with $\alpha_1 \neq \alpha$. Since $\mathrm{pr}_{(\alpha,H)}(\alpha_1, H_1) = (\mathrm{pr}_\alpha(\alpha_1), H_1)$ by Proposition 3.3(ii), it follows that $H' = H_1$, hence $H' \in J(F)$, as wanted.

Finally, part (d) is proved also by double inclusion. First, observe that $\Sigma(F)$ is irreducible by (b), so that it cannot be just $\{\alpha\}$ by Theorem 2.8(ii). Therefore, Proposition 3.3(iv) implies that $\Sigma(F'_0)$ contains the image of $\Sigma(F)$ under $\mathrm{pr}_\alpha$. Reciprocally, take any $\alpha' \in \Sigma(F'_0)$. As before, we can assume that there exists $H' \in J(F'_0)$ such that $(\alpha', H') = \mathrm{pr}_{(\alpha,H)}(\alpha_1, H_1)$ for some $(\alpha_1, H_1) \in \tilde{J}(F)$ with $\alpha_1 \neq \alpha$. Hence Proposition 3.3(ii) implies $\alpha' = \mathrm{pr}_\alpha(\alpha_1)$. Since obviously $\alpha_1 \in \Sigma(F)$, the result follows.                                                                                              □

REMARK 3.5. Exactly in the same way as in Proposition 3.1, one could construct from $F$ and a jumping pair $(\alpha, H)$ the Steiner bundle defined by $T'^* \to S^* \otimes H^0(\mathcal{O}_H(1))$. This time we get an $(s, t-1)$-Steiner bundle $F'$ on $H$ and the same results of this section hold by permuting the roles of $J(F)$ and $\Sigma(F)$. In particular, if $\tilde{J}(F)$ has the maximal dimension allowed by Theorem 2.8(i), then also $\tilde{J}(F')$ has the maximal dimension allowed by Theorem 2.8(i); and if $\tilde{J}(F')$ is irreducible, then $\Sigma(F) = \Sigma(F')$. We will not prove it, since it is done exactly in the same way.

Before stating and proving our main result, we include, for the reader's convenience, the following easy lemma about varieties of minimal degree that we will need. By variety of minimal degree we mean a nondegenerate irreducible variety in a projective space such that its degree minus its codimension is 1. We recall (see, for example, [**4**, Theorem 19.9]) that a smooth variety of minimal degree is either a quadric, a rational normal scroll (this includes the whole projective space and rational normal curves) or a Veronese surface in $\mathbb{P}^5$.

LEMMA 3.6. *Let $X \subset \mathbb{P}^N$ be a proper smooth irreducible projective variety that is cut out by quadrics. Assume that $X$ contains an $r$-dimensional linear subspace $\Lambda$ such*

*that the projection of $X$ from $\Lambda$ is a subvariety $X' \subset \mathbb{P}^{N-r-1}$ of minimal degree with $\dim X' = \dim X - r$. Then also $X$ is a variety of minimal degree.*

*Proof.* The inequality $\dim X' > \dim X - r - 1$ implies $X$ is not a cone with vertex $\Lambda$, so that there exists a point $x \in \Lambda$ such that the line spanned by $x$ and a general point of $X$ is not contained in $X$. Since $X$ is cut out by quadrics, such a line cannot be trisecant to $X$, and hence the projection from $x$ sends $X$ birationally to some $X_1 \subset \mathbb{P}^{N-1}$. Therefore both the degree and codimension of $X_1$ drop by one with respect to those of $X$ (recall that $x$ is, by hypothesis, a smooth point of $X$), and thus $X$ is a variety of minimal degree if and only if $X_1$ is.

On the other hand, if $\Lambda_1$ is the $(r-1)$-dimensional image of $\Lambda$, then $X'$ is the image of $X_1$ under the linear projection from $\Lambda_1$. Since $\dim X' = \dim X_1 - r$, this means that $X_1$ is a cone with vertex $\Lambda_1$. Hence now $X'$ has the same degree and codimension as $X_1$, so that $X_1$ is a variety of minimal degree because $X'$ is. As observed before, this completes the proof.    $\square$

THEOREM 3.7.    *Let $F$ be an $(s,t)$-Steiner bundle on $\mathbb{P}^n$ with $s \geqslant 2$ and such that $\tilde{J}(F)$ has dimension $t - n - s + 1$. Then $\tilde{J}(F)$ is irreducible and $F$ is one of the Schwarzenberger bundles of Examples 1.7, 1.8, 1.9 or 1.10.*

*Proof.* By Proposition 3.4, we can construct an $(s-1, t-1-\epsilon)$-Steiner bundle $F'_0$ such that $\tilde{J}(F'_0)$ has dimension $t - n - s + 1 - \epsilon$. In particular, $F'_0$ has the maximal dimension allowed by Theorem 2.8(i). Iterating this process $s - 2$ times, we arrive at a reduced $(2, t'')$-Steiner bundle $F''$ such that $\tilde{J}(F'')$ has dimension $t'' - n - s + 1$. Thus Theorem 2.8(iv) implies that $\tilde{J}(F'')$ is a smooth rational normal scroll in $\mathbb{P}^{t''-1}$. Since $\tilde{J}(F'')$ is irreducible, it follows from Proposition 3.4(iii) that also $\tilde{J}(F)$ is irreducible, that $\tilde{J}(F'')$ is the image of $\tilde{J}(F)$ under a series of $s - 2$ inner projections from different linear subspaces, and that $J(F) = J(F'')$. Since we know that $J(F'')$ is a rational normal scroll, also $J(F)$ is. Similarly (see Remark 3.5), we can produce from $F$ a reduced Steiner bundle $F'''$ on $\mathbb{P}^1$, so that it follows from Theorem 2.8(v) that $\Sigma(F) = \Sigma(F''')$ is a rational normal scroll. On the other hand, Lemma 3.6 implies that $\tilde{J}(F)$ is a variety of minimal degree. Using the classification of smooth varieties of minimal degree, we study separately each of the three possibilities for $\tilde{J}(F)$ (we do not consider the possibility of a quadric, since $\tilde{J}(F)$ has codimension $n + s - 2$, and this is one only in the case $n = 1, s = 2$, which is trivial by Theorem 2.8).

(i) If $\tilde{J}(F)$ is a rational normal curve (hence $t = n + s$) of degree $t - 1$, then necessarily $\tilde{J}(F'')$ is also a rational normal curve obtained from $\tilde{J}(F)$ by projecting from $s - 2$ points on it, so that $t'' = t - s + 2 = n + 2$. Therefore, Theorem 2.8(iv) says that $F''$ is the Schwarzenberger bundle of the triplet $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(n))$, and in particular $J(F'')$ is a rational normal curve of degree $n$. Since $J(F) = J(F'')$, it follows that $\pi_2^* \mathcal{O}_{\mathbb{P}^{n*}}(1) = \mathcal{O}_{\mathbb{P}^1}(n)$. On the other hand, the equality $\mathcal{O}_{\tilde{J}(F)}(1) = \mathcal{O}_{\mathbb{P}^1}(n + s - 1)$ implies $\pi_1^* \mathcal{O}_{\mathbb{P}(S)}(1) = \mathcal{O}_{\mathbb{P}^1}(s - 1)$. The fact that $\tilde{J}(F)$, $\Sigma(F)$ and $J(F)$ are rational normal curves implies that the hypotheses of Lemma 2.4(iii) are satisfied, so that we are in the case of Example 1.7 (of course, this is the case obtained in [**1**, **9**], because we are dealing with Steiner bundles of rank $n$).

(ii) If $\tilde{J}(F)$ is a Veronese surface, then $t - n - s + 1 = 2$ and $t = 6$. An inner projection produces a rational normal scroll only when projecting from one or two points, so that $s = 3, 4$. If $s = 4$, then $\tilde{J}(F'')$ is a smooth quadric in $\mathbb{P}^3$, so that $J(F'')$ is a line. Since $J(F'') = J(F)$ and there are no regular maps from the Veronese surface to $\mathbb{P}^1$, this case is not possible. Therefore $s = 3$ (hence $n = 2$) and $\tilde{J}(F'')$ is a cubic surface scroll in $\mathbb{P}^4$, so that $J(F'')$ is isomorphic to $\mathbb{P}^2$. Since the map $\pi_2 : \tilde{J}(F) \to J(F)$ has linear fibres, it follows that it is an isomorphism and $\pi_2^* \mathcal{O}_{\mathbb{P}^{n*}}(1) \cong \mathcal{O}_{\mathbb{P}^2}(1)$. And since the hyperplane class of $\tilde{J}(F)$ is $\mathcal{O}_{\mathbb{P}^2}(2)$, it also follows that

$\pi_1^* \mathcal{O}_{\mathbb{P}^{n*}}(1) \cong \mathcal{O}_{\mathbb{P}^2}(1)$ and $\pi_1$ is also necessarily an isomorphism. By Lemma 2.4(iii), we are in the case of Example 1.10.

(iii) Finally, assume that $\tilde{J}(F) \subset \mathbb{P}(T)$ is a rational normal scroll of dimension $t - n - s + 1 > 1$ (and degree $n + s - 1$). Since the only nontrivial splitting of the hyperplane section $h$ of $\tilde{J}(F)$ into two globally generated line bundles is

$$\mathcal{O}_{\tilde{J}(F)}(h) = \mathcal{O}_{\tilde{J}(F)}(rf) \otimes \mathcal{O}_{\tilde{J}(F)}(h - rf)$$

for some integer $r > 0$ (as usual, $f$ represents the fibre of the scroll), one of the factors must be $\pi_1^* \mathcal{O}_{\mathbb{P}(S)}(1)$ and the other one must be $\pi_2^* \mathcal{O}_{\mathbb{P}^{n*}}(1)$.

Assume, for example, $\pi_1^* \mathcal{O}_{\mathbb{P}(S)}(1) = \mathcal{O}_{\tilde{J}(F)}(rf)$ and $\pi_2^* \mathcal{O}_{\mathbb{P}^{n*}}(1) = \mathcal{O}_{\tilde{J}(F)}(h - rf)$. In this case, since $\tilde{J}(F)$, $\Sigma(F)$ and $J(F)$ are varieties of minimal degree, Lemma 2.4(iii) implies that $F$ is the Schwarzenberger bundle of the triplet $(\tilde{J}(F), \mathcal{O}_{\tilde{J}(F)}(rf), \mathcal{O}_{\tilde{J}(F)}(h - rf))$. Hence

$$s = h^0(\mathcal{O}_{\tilde{J}(F)}(rf)) = r + 1,$$
$$n + 1 = h^0(\mathcal{O}_{\tilde{J}(F)}(h - rf)) = t - r(t - n - s + 1)$$

so that $t - n - s + 1 = t - (t - r(t - n - s + 1) - 1) - (r + 1) + 1$ and thus $(r - 1)(t - n - s) = 0$, which implies $r = 1$, so that we are in the case of Example 1.9.

The case $\pi_1^* \mathcal{O}_{\mathbb{P}(S)}(1) = \mathcal{O}_{\tilde{J}(F)}(h - rf)$ and $\pi_2^* \mathcal{O}_{\mathbb{P}^{n*}}(1) = \mathcal{O}_{\tilde{J}(F)}(rf)$ is analogous, and we would obtain here Example 1.8. $\qquad\square$

If we just want to study the dimension of the set of jumping hyperplanes, we have the following.

COROLLARY 3.8. *Let $F$ be an $(s, t)$-Steiner bundle with $s \geqslant 2$. Then $J(F)$ has dimension at most $t - n - s + 1$, with equality if and only if $F$ is the Schwarzenberger bundle of one of the following triplets $(X, L, M)$:*
  (i) *$X = \mathbb{P}^1$, $L = \mathcal{O}_{\mathbb{P}^1}(s - 1)$, $M = \mathcal{O}_{\mathbb{P}^1}(n)$;*
  (ii) *$X \subset \mathbb{P}^{t-1}$ a smooth rational normal scroll of dimension $t - n - 1$ and degree $n + 1$ different from $\mathbb{P}^1 \times \mathbb{P}^n$ (that is, $t \neq 2n + 1$) and $L = \mathcal{O}_X(f)$, $M = \mathcal{O}_X(h - f)$ (see Example 1.9);*
  (iii) *$X = \mathbb{P}^2$, $L = M = \mathcal{O}_{\mathbb{P}^2}(1)$.*

*Proof.* The inequality follows from Theorem 2.8(i) using that $\dim J(F) \leqslant \dim \tilde{J}(F)$. In case of equality, we have to remove from Theorem 3.7 the cases in which $\dim J(F) < \dim \tilde{J}(F)$. Observe that the case $t = s + 1$ in Example 1.8 (that is, when $\dim J(F) = \dim \tilde{J}(F) = 1$) becomes the case $n = 1$ in Example 1.7, so that we do not need to consider it. $\qquad\square$

We also have this improvement of Re's results in the case of line bundles.

COROLLARY 3.9. *Let $L, M$ be two globally generated line bundles on an irreducible variety $X$, and assume that $L \otimes M$ is ample. If $W$ is the image of the multiplication map $H^0(L) \otimes H^0(M) \to H^0(L \otimes M)$, then $\dim(W) \geqslant h^0(L) + h^0(M) + \dim(X) - 2$, with equality if and only if the multiplication map is surjective and there is a triplet $(X', L', M')$ as in Examples 1.7, 1.8, 1.9 or 1.10 such that there exists a finite map $f : X \to X'$ satisfying $L = f^* L'$ and $M = f^* M'$.*

*Proof.* Let $F$ be the Schwarzenberger bundle of the triplet $(X, L, M)$. Then $\tilde{J}(F)$ is the image of $X$ via $L \otimes M$. Since $L \otimes M$ is ample and globally generated, it follows

that $\dim(\tilde{J}(F)) = \dim(X)$. As observed in Example 1.5, the reduced summand of $F$ is an $(h^0(L), \dim(W))$-Steiner bundle $F_0$, and obviously $\tilde{J}(F) = \tilde{J}(F_0)$. Thus the wanted inequality is just Theorem 2.8(i) when applied to $F_0$. In case we have equality, we know by Theorem 3.7 that $F$ is the Schwarzenberger bundle of a triplet $(X', L', M')$ as in Examples 1.7, 1.8, 1.9 or 1.10, and in particular $F$ is reduced, that is, $W = H^0(L \otimes M)$. Moreover, the proof gives that $X'$ is $\tilde{J}(F)$, that is, the image of $X$ via the map $f$ defined by $L \otimes M$. Also, since the composition $X \xrightarrow{f} X' \xrightarrow{\pi_1} \mathbb{P}(H^0(L))$ is the map defined by $L$, it follows that $L = f^* \pi_1^* \mathcal{O}_{\mathbb{P}(H^0(L))}(1) = f^* L'$, and similarly we obtain $M = f^* M'$. □

REMARK 3.10. It could seem *a priori* that it is possible to obtain Theorem 3.7 as a Corollary of the corresponding result of [**9**] for Steiner bundles of rank $n$. In fact, we can always take a general quotient $T \to T_1$ of dimension $n + s$ and, if $K$ is its kernel, we get a commutative diagram

$$
\begin{array}{ccccccccc}
& & 0 & & & & 0 & & \\
& & \downarrow & & & & \downarrow & & \\
& & K \otimes \mathcal{O}_{\mathbb{P}^n} & = & K \otimes \mathcal{O}_{\mathbb{P}^n} & & & \\
& & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & S \otimes \mathcal{O}_{\mathbb{P}^n}(-1) & \longrightarrow & T \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & F & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S \otimes \mathcal{O}_{\mathbb{P}^n}(-1) & \longrightarrow & T_1 \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & F_1 & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

in which now $F_1$ is a Steiner bundle of rank $n$. From this diagram, it is not difficult to see that $\tilde{J}(F_1)$ is the intersection of $\tilde{J}(F)$ with $\mathbb{P}(T_1)$. Since $\mathbb{P}(T_1)$ has codimension $t - n - s$ in $\mathbb{P}(T)$, it follows that $\dim \tilde{J}(F_1) \geqslant \dim \tilde{J}(F) - t + n + s$. Since the dimension of $\tilde{J}(F_1)$ is at most 1 (by Theorem 2.8(i), which is in this case the result of [**9**]), it follows that $\tilde{J}(F)$ has dimension at most $t - n - s + 1$. Moreover, if equality holds, we can apply the known result for $F_1$ and get that $\tilde{J}(F_1)$ is a rational normal curve, so that $\tilde{J}(F)$ has only one component of maximal dimension, which is a variety of minimal degree in $\mathbb{P}(T)$. However, such a proof does not exclude the possibility that $\tilde{J}(F)$ (or $J(F)$) has other components of smaller dimension, while our proof shows the irreducibility of $\tilde{J}(F)$. Hence our proof actually provides a positive answer to Question 0.2 for the Examples 1.7–1.10.

REMARK 3.11. The proof of Theorem 3.7 gives an idea of the difficulty of proving a similar result for arbitrary $a, b$. Independently of the fact that we were not able to find a reasonable bound for the dimension of $J_{a,b}(F)$, the main obstacle to prove something analogous to Theorem 3.7 is that we do not have a first induction step to apply an iteration using Proposition 3.1. Indeed, the minimal value of $s$ would be $s = a + 1$ (see Lemma 2.7), but as observed in Remark 2.6, a result like Theorem 2.8(iv) cannot hold because, for general values of $a, b$, one expects $\tilde{J}_{a,b}(F)$ to be empty, even for $s = a + 1$. The same problem remains when trying to apply the iteration process explained in Remark 3.5, since the first step should be a Steiner bundle on $\mathbb{P}^{b+1}$, for which we also expect $\tilde{J}_{a,b}(F)$ to be empty for general values of $a, b$.

On the other hand, it would also be nice to generalize Theorem 3.7 to arbitrary $a, b$ in order to generalize the improvement of Re's results given in Corollary 3.9 to arbitrary rank. Since our proof for $a = b = 1$ is closely related to the classification of varieties of minimal degree in the projective space, a generalization to arbitrary $a, b$ is likely to depend on a good theory of varieties of minimal degree in Grassmannians (see [**7**] for a first natural approach).

REMARK 3.12.   In [**8**], Soares gave a natural definition of Steiner bundle on any projective variety. It would be nice to have also the notion of Schwarzenberger bundle in her general context. For example, to get a natural definition on Grassmannians, one could take a triplet $(X, L, M)$ and fix an integer $r$ such that, for each $r$-dimensional subspace $V \subset H^0(M)$ the natural map $H^0(L) \otimes V \to H^0(L \otimes M)$ is injective. Let us consider $G = G(r, H^0(M))$, the Grassmann variety of linear subspaces of dimension $r$ in $H^0(M)$, and let $\mathcal{U}$ be the rank $r$ universal subbundle of $G$. Then there is an exact sequence of vector bundles on $G$:

$$0 \longrightarrow H^0(L) \otimes \mathcal{U} \longrightarrow H^0(L \otimes M) \otimes \mathcal{O}_G \longrightarrow F \longrightarrow 0$$

defining $F$ as a cokernel. This is a Steiner bundle on $G$ in the sense of [**8**], so that it seems natural to define Schwarzenberger bundles on $G$ as the bundles obtained in this way. Of course, when $r = 1$ we recover our definition of Schwarzenberger bundle on the projective space.

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